IDA

INSTITUTE FOR DEFENSE ANALYSES

Atmospheric Ray Tracing for Predicting Mirages

Irvin W. Kay

May 1995

Approved for public release;
distribution unlimited.

IDA Document D-1692

Log: H 95-046645

This work was conducted under contract DASW01 94 C 0054, DARPA Assignment A-180, for the Defense Advanced Research Projects Agency. The publication of this IDA document does not indicate endorsement by the Department of Defense, nor should the contents be construed as reflecting the official position of that Agency.

PREFACE

The Institute for Defense Analyses was requested to assist the Office of Naval Research in the planning and execution of the Infrared Analysis, Modeling, and Measurements Program (IRAMMP). This document summarizes a portion of the work performed with ONR under Task A-180, "Infrared Clutter Characterization and Modeling," on alternative algorithms for Infrared Search and Track (IRST) systems during the period June 1994 to September 1995. The work was performed in coordination with Mr. Douglas N. Crowder, Naval Surface Warfare Center (NSWC), for the Sensor Technology Office, Special Projects Office, ARPA, under the technical cognizance of Mr. Thomas Wiener.

This document has not been subjected to formal IDA review.

CONTENTS

FIGURES

I. INTRODUCTION

From the Earth's surface to altitudes that are not too high, the temperature may decrease with increasing height. If the temperature decreases at a large enough rate, the index of refraction will increase with increasing height.[1] Such behavior can cause a mirage, i.e., more than one image of an object located at certain distances from the viewer.

This particular version of the effect is called an "inferior" mirage because the normal image (one due to a bundle rays, each of which connects an object point to the viewing eye without crossing another ray in the bundle) appears to be above a second, abnormal, image. The phenomenon occurs when some rays emanating from an object toward the Earth's surface either begin with or achieve negative curvature before reaching the surface, so that they bend upward, being totally reflected by the atmosphere. These rays then cross one another, forming a caustic, before reaching the viewer. The result is an inverted image of the object appearing below the normal image created by other rays from the same object points. Each ray contributing to the inferior mirage image passes through a minimum height between the object and the eye, but no rays contributing to the normal image have minima between the object and the eye.

Another version, called a "superior" mirage, exists when an abnormal image appears above the normal one and is the result of rays that are totally reflected from atmospheric layers at higher altitudes than the viewing eye. The superior mirage is due to a temperature inversion that occurs when the Earth's surface, which in this case is often water or ice, is colder than the air above it, and the atmospheric temperature increases with height instead of decreasing. Atmospheric refraction then causes light rays to bend with positive curvature, i.e., in the same sense as the Earth's surface. Rays that arrive from a sufficiently distant object converge on the viewing eye from above, each attaining a maximum altitude between the eye and the object.

If the rays intersect one another, forming a caustic, so that they arrive at the eye in reverse order with respect to height, the image that they form of the object will appear to be

---

[1] However, the rate at which the temperature decreases, the so-called lapse rate, will normally grow smaller with increasing height, eventually becoming small enough for the index of refraction to reach a maximum and then decrease with increasing height (cf. Ref. 1, pp. 77 ff.).

inverted. Other rays, arriving at the eye with smaller elevation angles, will have no maxima between the eye and the object and will therefore maintain their initial order with respect to height. The two sets of rays will then form an inverted, superior mirage image above a true image of the object.

As an aid to understanding some of the observed behavior of mirages, this document considers some analytical connections between geometrical properties of such phenomena and physical properties of the atmosphere. Section II reviews the well known relationship between the index of refraction at optical-infrared frequencies and the atmospheric temperature as functions of altitude. Section III considers the geometrical properties of rays in a spherically stratified, continuous propagation medium, assumed to be a good model of the Earth's lower atmosphere. Section IV discusses consequences of this analysis for the nature of complex mirages.

II. INDEX OF REFRACTION IN A SPHERICALLY SYMMETRIC ATMOSPHERE

It is customary to deal with the index of refraction n(r) of the atmosphere in terms of the refractive modulus (or N unit) defined by

$$N = (n - 1) \times 10^6 \quad . \tag{1}$$

For optical wavelengths greater than 0.23 μ and for infrared wavelengths, N for the atmosphere is given by Edlen's formula (Ref. 2, p. 18-7)

$$N = N_d - N_w \quad , \tag{2}$$

where the first term on the right side of (2) is for a dry atmosphere and is given by

$$N_d = \left[a_0 + \frac{a_1}{1 - \left(\frac{\nu}{b_1}\right)^2} + \frac{a_2}{1 - \left(\frac{\nu}{b_2}\right)^2} \right] \frac{P}{P_0} \frac{(T_0 + 15.0)}{T} \quad , \tag{3}$$

and the second term, which is the contribution due to water vapor, by

$$N_w = \left[c_0 - \left(\frac{\nu}{c_1}\right)^2 \right] \frac{P_w}{P_0} \quad . \tag{4}$$

In (3) and (4) P is the total pressure in mb, T is the temperature in °K, P_0 = 1013.25 mb, T_0 = 273.15 °K, P_w is the partial pressure of water vapor in mb, and $\nu = 10^4/\lambda$ which is the wave number in cm^{-1} for the wavelength λ in microns. Also

$$a_0 = 83.42,$$
$$a_1 = 185.08,$$
$$a_2 = 4.11,$$
$$b_1 = 1.14 \times 10^5,$$
$$b_2 = 6.24 \times 10^4,$$
$$c_0 = 43.49,$$
$$c_1 = 1.7 \times 10^4.$$

Numerical examples assuming a value of 25 mb for P_w and ground temperatures on the order of 300 °K indicate that the contribution of the water vapor term N_w to the refractive modulus is well below 1 percent of its total value and can therefore be neglected.

It is clear from (3) and (4) that the behavior of the modulus of refraction N as a function of r is the result of the functional dependence on r of the temperature T, the total pressure P, and the water vapor partial pressure P_w. If the water vapor contribution is neglected the functions P(r) and T(r) will determine the atmospheric variation of the index of refraction, by means of which the geometrical properties of optical rays in the lower atmosphere can be determined.

The hydrostatic relation between the pressure P and atmospheric density ρ as functions of the radial distance r in an Earth-centered polar coordinate system is

$$\frac{dP}{dr} = -g\rho \quad , \tag{5}$$

where g is the acceleration of gravity. The gas law for an ideal gas is

$$P = RT\rho \quad , \tag{6}$$

where T is the temperature and R is the gas constant. Eliminating ρ from (5) and (6) leads to the differential equation

$$\frac{dP}{dr} = -\frac{gP}{RT} \quad , \tag{7}$$

for which the solution is

$$P = P_0 e^{-\frac{g}{R}\int_{r_e}^{r} \frac{dz}{T}} \quad , \tag{8}$$

where r_e is the radius of the Earth and P_0 is the atmospheric pressure at the Earth's surface.

In (8) g = 9.8 m/sec^2. Also, the gas constant R in erg/°K-mol is 8.3145×10^7, and the molar weight of air is .028971 kgm (cf. Ref. 3, p.17); therefore, since 1 Joule = 10^7 erg, in the MKS system

$$R = 286.99389 \text{ Joule/°K-kgm} \quad . \tag{9}$$

Neglecting the contribution N_w due to water vapor, it follows from (2), (3), and (8) that the refractive modulus is given by

$$N = \frac{78580}{T(r)} e^{-.03417 \int_{r_e}^{r} \frac{dz}{T(z)}} \quad . \tag{10}$$

Then, according to (1),

$$n(r) = 1 + \frac{.07858}{T(r)} e^{-.03417 \int_{r_e}^{r} \frac{dz}{T(z)}} \quad . \tag{11}$$

III. RAYS IN A SPHERICALLY SYMMETRIC MEDIUM

For most purposes it can be assumed that the Earth is a sphere and that the index of refraction profile of the atmosphere is continuously stratified in concentric spherical layers down to the Earth's surface. A ray can be represented in polar coordinates relative to an origin at the center of the Earth by an equation of the form

$$r = r(\theta) \quad . \tag{12}$$

It is assumed that the index of refraction of the atmosphere is a function n(r) relative to this polar coordinate system. Then the function $r(\theta)$ representing a ray satisfies (cf. Ref. 4, p. 123)

$$\theta - \theta_0 = \alpha \int_{r_0}^{r(\theta)} \frac{d\rho}{\rho\sqrt{\rho^2 n^2(\rho) - \alpha^2}} \quad . \tag{13}$$

The one parameter family of rays determined, for different values of α, by (13) all pass through the point (r_0, θ_0). For a given ray the corresponding parameter α can be expressed in terms of the angle ψ between the radius vector from the origin of the coordinate system to (r_0, θ_0) and the tangent to the ray at that point by[2]

$$\alpha = \pm r_0 n(r_0) \sin\psi \quad \cdot$$

so that (13) becomes

---

2 This is a result of the relation $\cot\psi = \left.\frac{1}{r_0}\frac{dr}{d\theta}\right|_{\theta=\theta_0}$ (cf. Ref. 5, p. 265).

$$\theta - \theta_0 = \pm r_0 n(r_0) \sin\psi \int_{r_0}^{r} \frac{d\rho}{\rho\sqrt{\rho^2 n^2(\rho) - r_0^2 n^2(r_0)\sin^2\psi}} \quad . \tag{14}$$

If it is contributing to a mirage, a ray must pass through a maximum or minimum height above the Earth's surface; i.e., at some angle θ_m the radial coordinate of a point on the ray must satisfy the relation

$$\left.\frac{dr}{d\theta}\right|_{\theta=\theta_m} = 0 \quad .$$

This condition applied to (13) leads to

$$\left.\frac{dr}{d\theta}\right|_{\theta=\theta_m} = \pm \frac{r_m}{r_0 n(r_0)\sin\psi} \sqrt{r_m^2 n^2(r_m) - r_0^2 n^2(r_0)\sin^2\psi} = 0 \quad ,$$

which is equivalent to

$$r_m n(r_m) = r_0 n(r_0) \sin\psi \quad . \tag{15}$$

Since (15) implies that $r_m n(r_m)$ cannot be larger than $r_0 n(r_0)$, if r_m is a maximum for the ray, n(r) must decrease between r_0 and r_m. If r_m is a minimum for the ray, and therefore no larger than r_0, n(r) must be an increasing function of r over some neighborhood of r_m that may or may not include r_0. In the rest of this document it will be assumed that this is the case for r values within some layer at the Earth's surface.[3]

The curvature of a curve expressed in polar coordinates, e.g., the ray given by (13), is given by (cf. Ref. 5, p. 291)

---

3 An example, discussed in Ref. 1, is the case of the unstable layer that usually forms on the ground after daybreak and disappears at night.

$$K = \frac{r^2 + 2\left(\frac{dr}{d\theta}\right)^2 - r\frac{d^2 r}{d\theta^2}}{\left[r^2 + \left(\frac{dr}{d\theta}\right)^2\right]^{\frac{3}{2}}} \quad . \tag{16}$$

Using the relations

$$\frac{dr}{d\theta} = \pm \frac{r_m}{r_0 n(r_0)\sin\psi}\sqrt{r^2 n^2(r) - r_0^2 n^2(r_0)\sin^2\psi} \tag{17}$$

and

$$\frac{d^2 r}{d\theta^2} = \frac{dr}{d\theta}\frac{d}{dr}\left(\frac{dr}{d\theta}\right) \quad , \tag{18}$$

the ray curvature given by (14) and (16) becomes

$$K = -\frac{r_0 n(r_0)}{r n^2(r)}\frac{dn}{dr}\sin\psi \quad . \tag{19}$$

It follows from (19) that the ray curvature is negative where the slope of the index of refraction is positive and positive where the slope is negative.

Using (19) with (11), it is possible to obtain the curvature of a ray at any point in terms of the temperature profile. A straightforward calculation provides the result:

$$K(r) = \frac{[n(r) - 1] r_0 n(r_0)\sin\psi}{r n^2(r)}\left(\frac{dT}{dr} + 0.03417\right) \quad . \tag{20}$$

It is evident from (20) that at a given altitude the curvature of a ray is positive if the slope of the temperature profile is greater than – .03417, is zero if the slope is equal to – .03417, and is negative if the slope is less than – .03417.

Suppose that the eye level of the viewer is at the radial distance r_0 from the Earth's center and the radial line is the polar axis $\theta = 0$. Also, suppose that a ray begins at the point $(r_0,0)$ with the direction of its tangent there specified by an obtuse zenith (measured from the polar axis) angle ψ. Then the ray will be given by (14) and (15) in the form

$$\theta = r_m n(r_m) \int_r^{r_0} \frac{d\rho}{\rho\sqrt{\rho^2 n^2(\rho) - r_m^2 n^2(r_m)}}, 0 \le \theta \le \theta_m \quad , \tag{21}$$

for $r \ge r_m$, which is the minimum radial coordinate of the ray as determined by (15), and by

$$\theta = \theta_m + r_m n(r_m) \int_{r_m}^{r} \frac{d\rho}{\rho\sqrt{\rho^2 n^2(\rho) - r_m^2 n^2(r_m)}}, 2\theta_m \ge \theta \ge \theta_m \quad , \tag{22}$$

for $r \le r_0$. In (22) θ_m is the angular coordinate of the ray point whose radial coordinate is r_m.[4]

The astronomical horizon, defined as the direction orthogonal to a plumb line, is given by $\psi = \frac{\pi}{2}$. A declination angle ϕ from the astronomical horizon is therefore determined by $\psi = \phi + \frac{\pi}{2}$, so that for a small declination

$$\sin\psi = \sin\left(\phi + \frac{\pi}{2}\right) = \cos\phi \sim 1 - \frac{\phi^2}{2} \quad . \tag{23}$$

[4] Of course, if r_m is smaller than the Earth's radius, the ray will end at the Earth's surface and will have no physical minimum.

An optical horizon is defined by a ray whose minimum is at the Earth's surface and is therefore determined by an initial declination satisfying (15) with r_m set equal to the Earth's radius r_e. An approximation to the declination angle of a horizon ray follows from (23):

$$\phi \sim \sqrt{2\left\{\frac{r - r_e}{r_e} + 10^{-6}\left[N(r_0) - N(r_e)\right]\right\}} \quad , \tag{24}$$

where the function N(r) is the modulus of refraction.

Given that the ray curvature K(r) is zero at r_K, at any point (r_m,θ_m) for which the radial coordinate satisfies (15), whether r_m is a minimum or a maximum will depend on whether $r_m \le r_K$ or $r_m > r_K$. In particular, if ψ is a right angle the radial coordinate r_0 of the ray associated with ψ will be a minimum or a maximum at the point $(r_0,0)$, depending on whether $r_0 \le r_K$ or $r_0 > r_K$.[5]

If $r_0 > r_K$, when $\psi \ge \frac{\pi}{2}$ the ray will start at $(r_0,0)$ with positive curvature and will approach the Earth's surface with the index of refraction n(r) increasing as r decreases. When the radial coordinate of the ray decreases below r_K at which n(r) reaches a maximum, the index decreases with r until the ray either reaches the Earth's surface or passes through a point (r_m,θ_m) where the radial coordinate is a minimum. Up to this point (21) determines the ray. As the angle θ increases beyond θ_m, (22) determines the behavior of the ray until its radial coordinate reaches a maximum.

If $\psi < \frac{\pi}{2}$, instead of (21) the ray equation becomes

$$\theta = r_m n(r_m) \int_{r_0}^{r} \frac{d\rho}{\rho\sqrt{\rho^2 n^2(\rho) - r_m^2 n^2(r_m)}}, 0 \le \theta \le \theta_m \quad , \tag{25}$$

for $r_0 \le r \le r_m$, where at the ray point (r_m,θ_m), the radial coordinate is a maximum. Beyond this point (22) determines the behavior of the ray as θ increases until the ray

---

[5] If $r_0 = r_K$, whether the ray has a minimum or a maximum radial coordinate at $(r_0,0)$ will depend on the second, or perhaps even a higher order, derivative of n(r).

reaches the Earth's surface or achieves a minimum radial coordinate. If in (22) ψ equals the angle ψ_m that satisfies the equation

$$\sin\psi_m = \frac{r_e n(r_e)}{r_0 n(r_0)} \sim \frac{n(r_e)}{n(r_0)} \quad , \tag{26}$$

then according to (15) the ray will become tangent to the Earth's surface and therefore will determine an optical horizon for the viewer.

To examine the effect of atmospheric refraction on images of objects located at distances much smaller than the Earth's radius, it is useful to employ approximate versions of equations (21) and (22) expressed in Cartesian coordinates. The origin of the Cartesian coordinate system is on the Earth's surface, so that the vertical (z) axis is coincident with the $\theta = 0$ line, and the horizontal (x) axis is tangent to the Earth. The (x,z) coordinates then may be related to the (r,θ) by

$$\begin{aligned} x &= r\theta \quad , \\ z &= r - r_e \quad . \end{aligned} \tag{27}$$

The ray equations given by (21) or (25) in polar coordinates take the form

$$x = \pm(z+r_e)(z_m+r_e)\nu(z_m)\int_{z_0}^{z} \frac{d\zeta}{(\zeta+r_e)\sqrt{(\zeta+r_e)^2\nu^2(\zeta)-(z_m+r_e)^2\nu^2(z_m)}} \quad , \tag{28}$$

where $\nu(z) = n(z+r_e)$ and $z_m = r_m - r_e$. Dropping terms in (28) that are small compared to the Earth's radius r_e, (28) becomes

$$x = \pm n(z_m)\int_{z_0}^{z} \frac{d\zeta}{\sqrt{n^2(\zeta)-n^2(z_m)}} \quad , \tag{29}$$

where n(z) is now used in place of ν(z) to represent the index of refraction in a plane stratified medium. In fact, (29) is the exact equation for a ray in a plane stratified medium.[6]

Starting with (29), a straightforward calculation of the ray curvature leads to

$$K = -\frac{n(z_m)}{n^2(z)}\frac{dn(z)}{dz} \tag{30}$$

instead of (19).

It is customary to modify the index of refraction n(z) to accommodate the fact that although rays in an atmosphere with a constant index of refraction n(r) relative to an Earth-centered polar coordinate system are straight lines, and a horizon ray from the eye is tangent to the Earth's surface, this cannot be the case for a flat Earth relative to a Cartesian coordinate system. The solution is to change n(z) from a constant to a variable index of refraction such that the corresponding horizon ray will curve enough to be tangent to the flat Earth, i.e., by retaining a higher order term in the approximation leading to (29).

A simple way to accomplish this is to start by getting rid of the integrals in (28) and (29). Differentiating both sides of (28) leads to

$$\begin{aligned}
\frac{dx}{dz} &= \pm\frac{x}{r_e+z} \pm \frac{(r_e+z_m)\nu(z_m)}{\sqrt{(r_e+z)^2\nu(z)^2-(r_e+z_m)^2\nu^2(z_m)}} \\
&= \pm\frac{x}{r_e+z} \pm \frac{\left(1+\frac{z_m}{r_e}\right)\nu(z_m)}{\sqrt{\left(1+\frac{z}{r_e}\right)^2\nu^2(z)-\left(1+\frac{z_m}{r_e}\right)^2\nu^2(z_m)}} \\
&\sim \pm\frac{x}{r_e+z} \pm \frac{\left(1+\frac{z_m}{r_e}\right)\nu(z_m)}{\sqrt{\nu^2-\nu^2(z_m)+\frac{2}{r_e}\left[z\nu^2(z)-z_m\nu^2(z_m)\right]}} \quad .
\end{aligned} \tag{31}$$

6 Cf. Ref. 6, p. 38, specialized to a plane y = 0.

Assuming that $\nu(z) \sim \nu(z_m)$, it follows from (31) that

$$\frac{dz}{dx} \sim \pm \frac{\sqrt{\frac{2}{r_e}(z - z_m)}}{1 + \frac{z_m}{r_e}} \quad . \tag{32}$$

Differentiating both sides of (29) and solving for n(z) leads to

$$n(z) = n(z_m)\sqrt{1 + \left(\frac{dz}{dx}\right)^2} \quad . \tag{33}$$

Substituting from (32) into (33) and assuming that $n(z_m) = \nu(z_m)$ provides the result

$$n(z) \sim \nu(z_m)\frac{\sqrt{1 + 2\frac{z_m}{r_e} + \frac{2(z-z_m)}{r_e}}}{1 + \frac{z_m}{r_e}} = \nu(z_m)\frac{\sqrt{1 + \frac{2z}{r_e}}}{1 + \frac{z_m}{r_e}} \sim \nu(z_m)\left(1 + \frac{z}{r_e}\right)\left(1 - \frac{z_m}{r_e}\right) \quad ,$$

so that

$$n(z) \sim \nu(z_m)\left(1 + \frac{z - z_m}{r_e}\right) \sim \nu(z_m) + \frac{z - z_m}{r_e} \sim \nu(z) + \frac{z - z_m}{r_e} \quad . \tag{34}$$

N.B., the correction to be applied to the index of refraction of a plane-stratified medium when using a flat Earth approximation is ray dependent: it depends on the height z_m at which the ray has a maximum or minimum. The usual correction

$$n(z) \sim \nu(z) + \frac{z}{r_e} \tag{35}$$

is valid only for a horizon ray, which has a minimum at the Earth's surface where $z_m = 0$.

The temperature profile depicted in Figure 1 as a function of the altitude z above the Earth's surface will lead to an index of refraction profile given by (11) in the Earth-centered polar coordinate system. When the flat Earth approximation is used, and the resulting index of refraction is modified in accordance with (35), the corresponding index of refraction in a Cartesian coordinate system with its origin on the surface of the Earth has the profile shown in Figure 2.

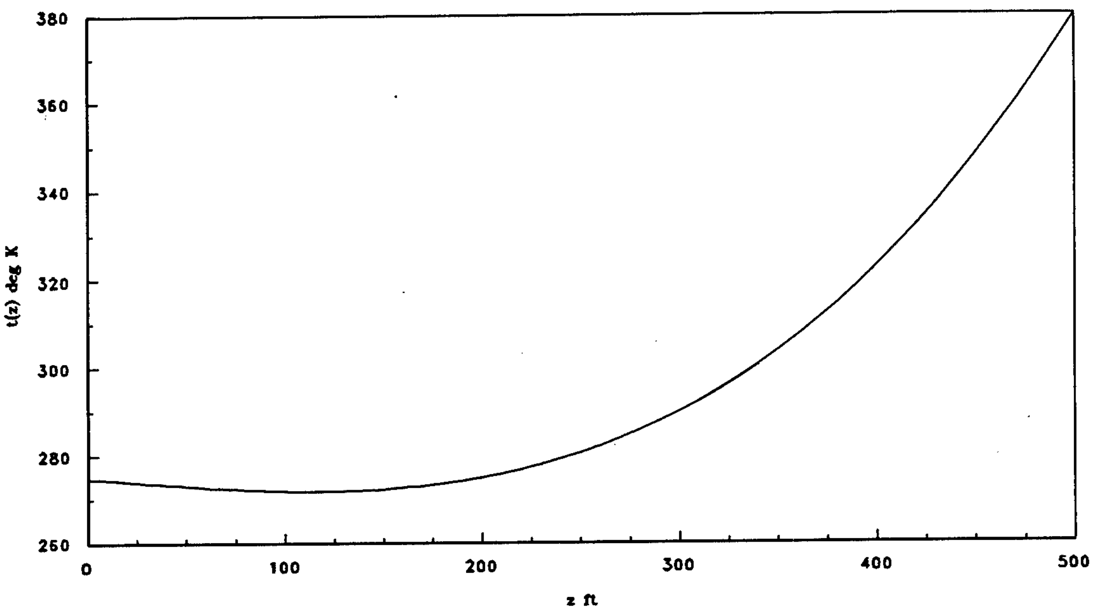


Figure 1. Temperature Profile

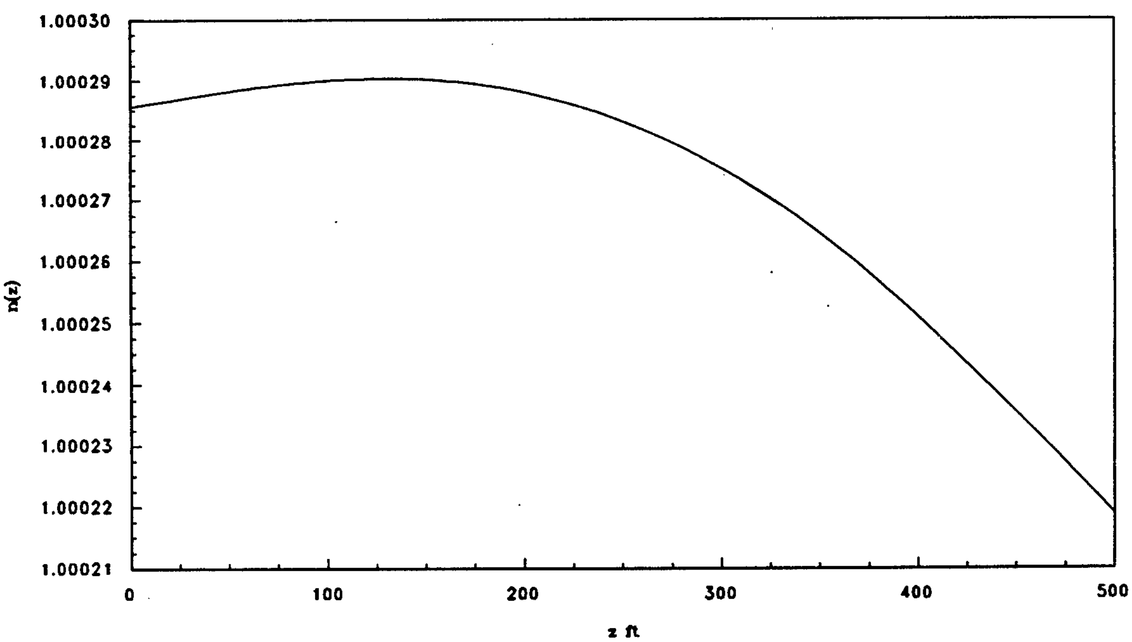


Figure 2. Index of Refraction Profile

IV. MIRAGE CONDITIONS

According to (15), in the Earth-centered polar coordinate system, for a ray passing through the eye at $(r_0,0)$ to have maxima or minima at several different heights r_i, the product $r_i n(r_i)$ must be the same at each height. With the corresponding Cartesian coordinate system and the flat Earth approximation, for which (29) provides the associated ray equation, the index of refraction $n(z_i)$ must have the same value at each height z_i where the ray maxima and minima occur.

This implies that $n(z)$ must have a minimum or a maximum between every two adjacent heights z_i and z_{i+1} where a ray minimum or maximum occurs. Moreover, where $n(r)$ or $n(z)$ has a minimum or maximum it follows from (19) or (30) that the ray curvature is zero. In either case, as the ray goes through the height where such a minimum or a maximum occurs, its curvature changes sign.

In addition, it is evident from (29) that if $n(z)$ is decreasing with height at the eye height z_0, the height z_m of a ray maximum or minimum closest to the eye along the ray path must be greater than z_0. Similarly, if $n(z)$ is increasing with height at z_0, z_m must be less than z_0.

Figures 3 and 4, along with Figure 2, illustrate these facts. Figure 3, which depicts an inferior mirage, shows a bundle of rays entering a viewing eye at a height of 200 feet, which is above the height where the associated index of refraction profile, shown in Figure 2, has a maximum. Figure 4, which depicts a superior mirage, shows a bundle of rays entering a viewing eye at a height of 80 feet, which is below the height where the profile has a maximum.

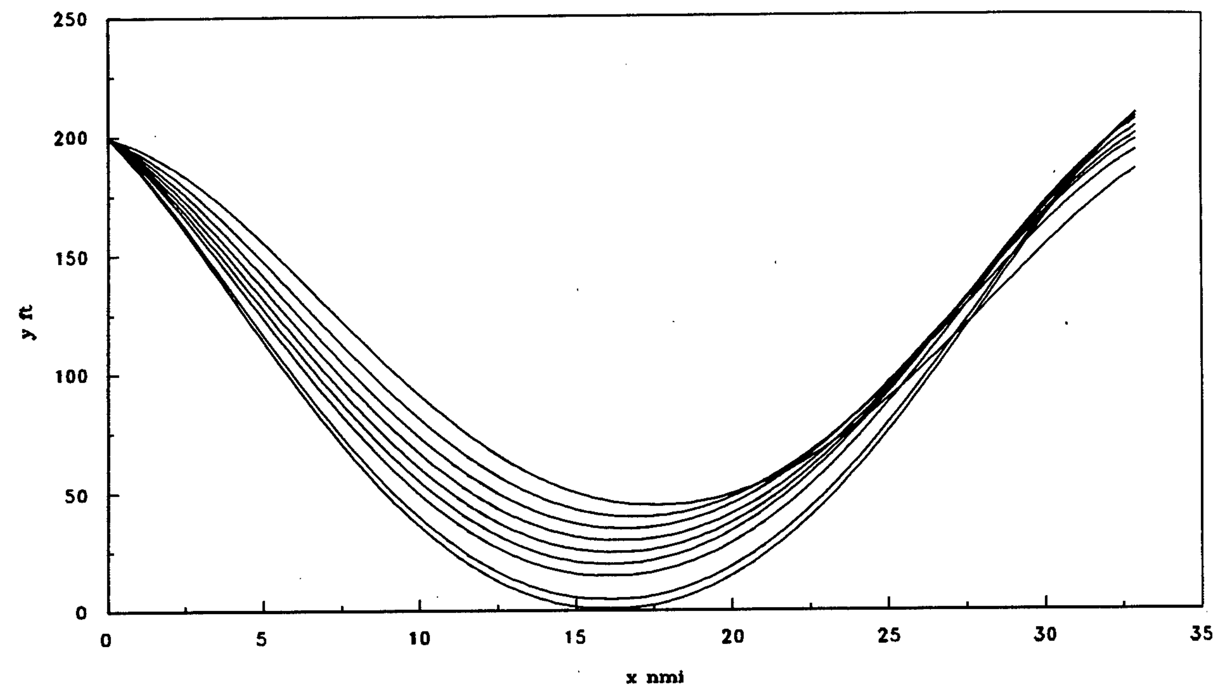


Figure 3. Inferior Mirage Rays

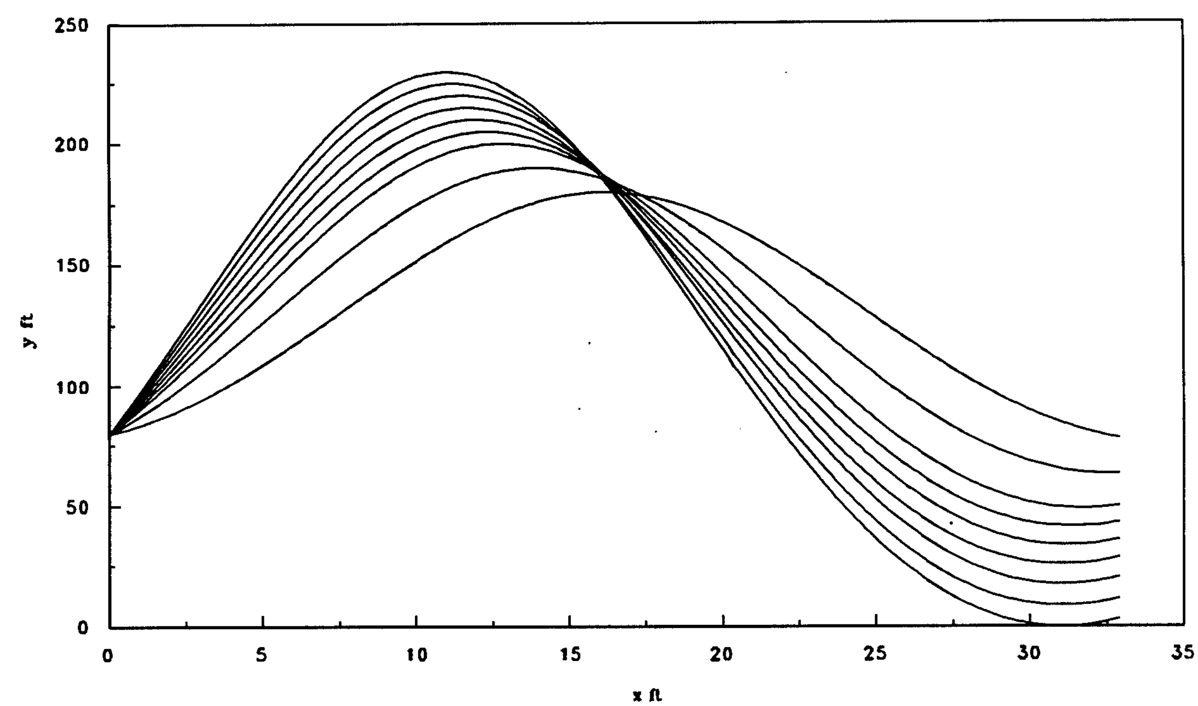


Figure 4. Superior Mirage Rays

REFERENCES

1. R. Geiger, *The Climate Near the Ground*, Harvard University Press, Cambridge, Mass., 1965.

2. A.S. Jursa (Ed.), *Handbook of Geophysics and the Space Environment*, Air Force Geophys. Lab., AFSC, USAF, 1985.

3. D.L. Book, *NRL Plasma Formulary*, Naval Research Lab., Washington, D.C., 1987.

4. M. Born and E. Wolf, *Principles of Optics* (Third Ed.), Pergamon Press, Oxford, England, 1965.

5. R. Courant, *Differential and Integral Calculus*, Volume I (Second Ed.), Interscience, New York, N.Y., 1937.

6. R.K. Luneberg, *Mathematical Theory of Optics*, University of California Press, Berkeley, Calif., 1964.

APPENDIX

This Appendix considers the numerical problem of calculating an atmospheric temperature profile for which the associated index of refraction has a maximum, as well as rays with specified maximum or minimum heights in the medium. The approach is aimed at using Mathcad Plus 5.0 software for Windows as the means of implementing the calculations.

The formulation is based on assuming that a cubic polynomial with only one real zero defines the temperature profile. Four parameters, one of which can be identified as a scale factor, define the polynomial, which has the form

$$t(z) = \left(\frac{z}{sf}\right)^3 - 2b\left(\frac{z}{sf}\right)^2 + \frac{\left(b^2 + c^2\right)}{sf} z + d \quad , \tag{A-1}$$

where a, b, c are arbitrary parameters and sf is the scale factor. The associated index of refraction n(z) is given by (11).

The relation (29) determines a ray with a maximum or a minimum at the height z_m and entering the eye at the height z_0. If a ray has a maximum or a minimum the horizontal Cartesian coordinate x must be a multi-valued function of the vertical coordinate z, and, as discussed in Section III for the case of a spherically symmetric medium, continuing x beyond the value given by (29) after z reaches the value z_m requires changing the integration interval and adding a constant. For programming the calculation in Mathcad, a somewhat more convenient approach is to solve a differential equation for the ray, obtaining z as a (single valued) function of x.

Differentiating both sides of (29) provides a suitable differential equation

$$\frac{dz}{dx} = \pm \frac{\sqrt{n^2(z) - n^2\left(z_m\right)}}{n\left(z_m\right)} \quad . \tag{A-2}$$

The right side of (A-2) must be positive if z is increasing and negative if z is decreasing. Therefore, if the critical point on the ray nearest the initial value is a maximum,[7] the sign must be positive, but if it is a minimum the sign must be negative. After integrating (A-2) from the eye position at the point $(0,z_0)$ to the nearest critical point, where the right side becomes infinite, the sign must change. If the integration is continued the sign must change after each critical point reached by the process.

Two versions of the Mathcad ray calculation spread sheet, based on solving the differential equation (A-2), using (A-1) for the temperature profile, follow. One, called raycalc3.mcd, assumes that the critical point on the ray nearest the eye level z_0 is a maximum; the other, which assumes that the nearest critical point is a minimum, is raycalc4.mcd. Both programs calculate Cartesian coordinates of points along a ray and store the data in a file. Each also plots the corresponding ray using the data created in this manner, and a later part of each program can plot nine rays in a single figure after the necessary data files have been created by running the earlier part.

---

7 The critical point may occur at z_m or at another value of z at which the index of refraction n(z) equals $n(z_m)$.

RAYCALC3.MCD

CAUTION! Click on Automatic in the Math menu to toggle to the manual mode before proceeding.

This program generates a temperature profile and the corresponding index of refraction profile. The function chosen for this purpose is a cubic polynomial that has only 1 real zero and depends on three independent parameters and a scale factor sf. The program also traces a ray, given its maximum height, but it assumes a ray with at most one maximum.

$$TOL := 10^{-8}$$

$$sf := 100 \quad b := 0 \quad c := 2 \quad d := 275 \quad a := b^2 + c^2 \quad b2 := \frac{2 \cdot b}{sf^2} \quad sf3 := sf^3 \quad a2 := -\frac{a}{sf}$$

$$t(x) := \frac{x^3}{sf3} - b2 \cdot x^2 + a2 \cdot x + d \qquad f := \frac{3}{sf3} \qquad g := 2 \cdot b2 \qquad h := 2 \cdot f$$

Check the lowest temperature, which is intitially in deg K., in deg C and deg F:

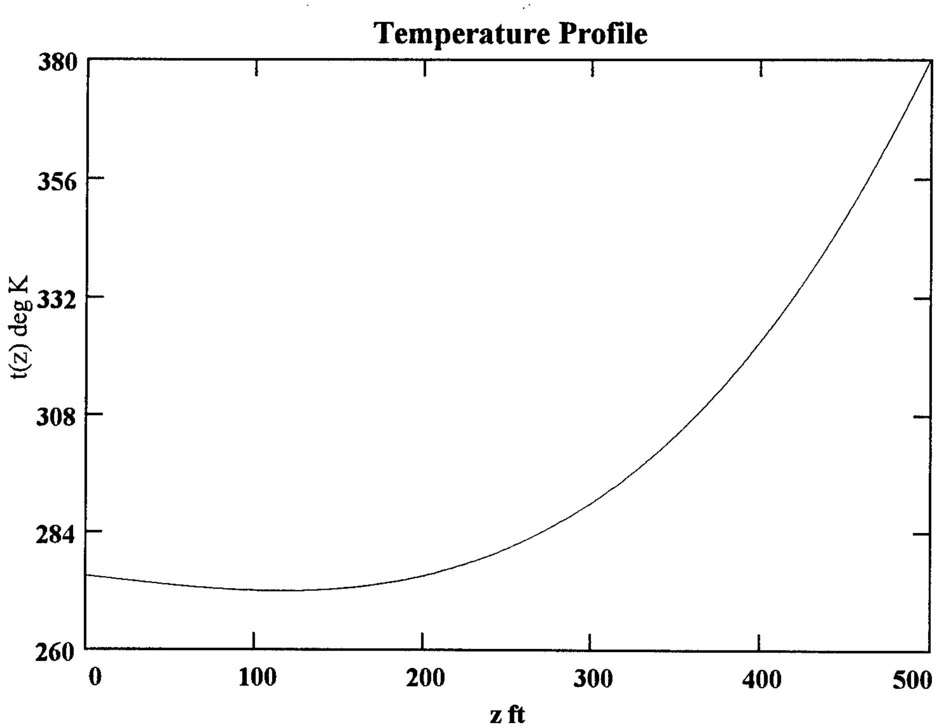
$$C := t(0) - 273 \quad C = 2 \qquad 9 \cdot \frac{C}{5} + 32.2 = 35.8 \qquad C := t(450) - 273$$

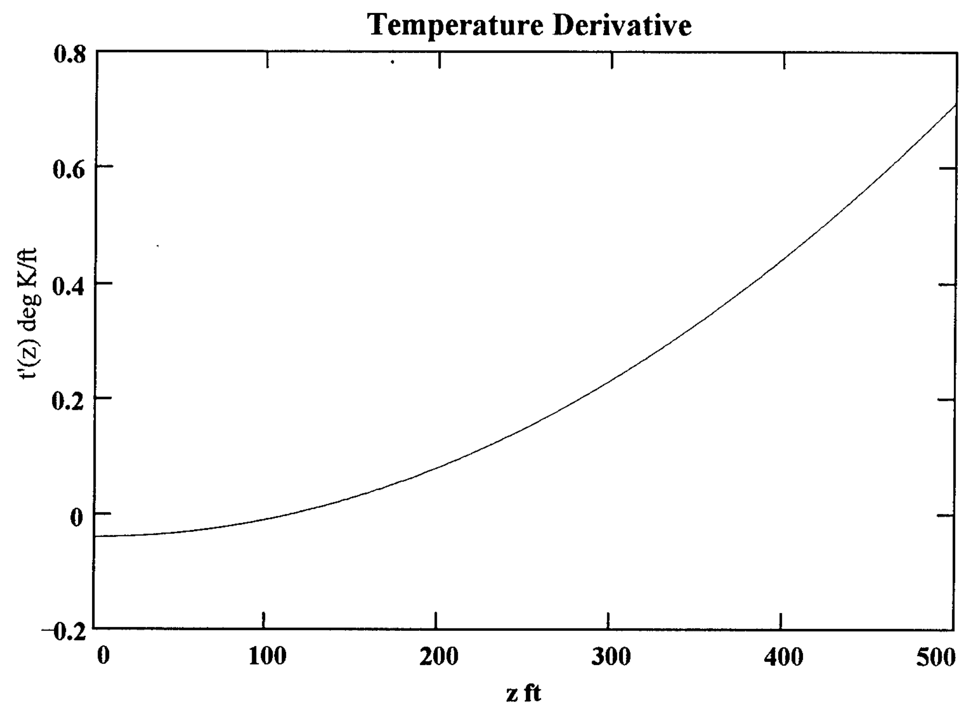
$$9 \cdot \frac{C}{5} + 32.2 = 167.425$$

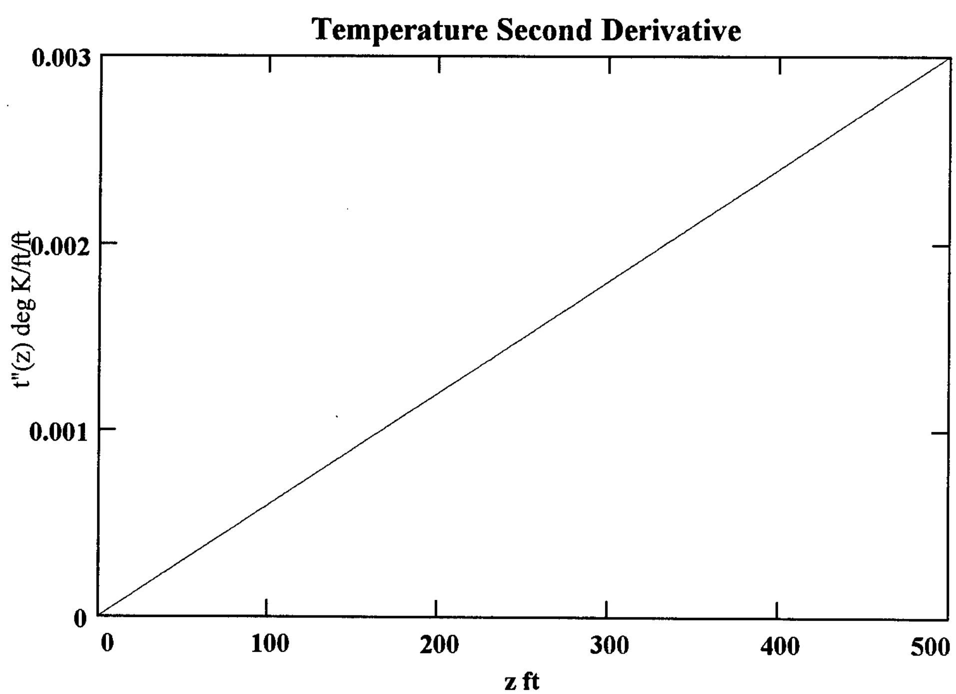
Get first and second derivatives t'(x) and t"(x) of the temperature profile.

$$t'(x) := f \cdot x^2 - g \cdot x + a2 \qquad t''(x) := h \cdot x - g$$

$$z := 0, 1 .. 500$$




Temperature Derivative
0.8
0.6
0.4
0.2
0
−0.2
t'(z) deg K/ft
0
100
200
300
400
500
z ft


Temperature Second Derivative
0.003
0.002
0.001
0
t''(z) deg K/ft/ft
0
100
200
300
400
500
z ft

Get the index of refraction profile. First define some universal constants: $N_0 := 77.5256$,

$\Gamma := .03417$, the standard atmospheric pressure at sea level $P_0 := 1013.25$, which, along

with the temperature profile t(z), determine the atomospheric pressure p(z) and the modulus of refraction N(z) as a function of the height z. The index of refraction n(z) is defined in terms of the modified modulus M(z). The parameter κ is .048 if the unit of height is feet or .157 if it is meters.

$$p(z) := P_0 \cdot e^{\left[-\Gamma \cdot \int_0^z \frac{1}{t(\zeta)} d\zeta\right]} \qquad N(z) := N_0 \cdot \frac{p(z)}{t(z)}$$

$$\kappa := .048 \quad M(z) := N(z) + \kappa \cdot z \quad n(z) := 1 + 10^{-6} \cdot M(z)$$

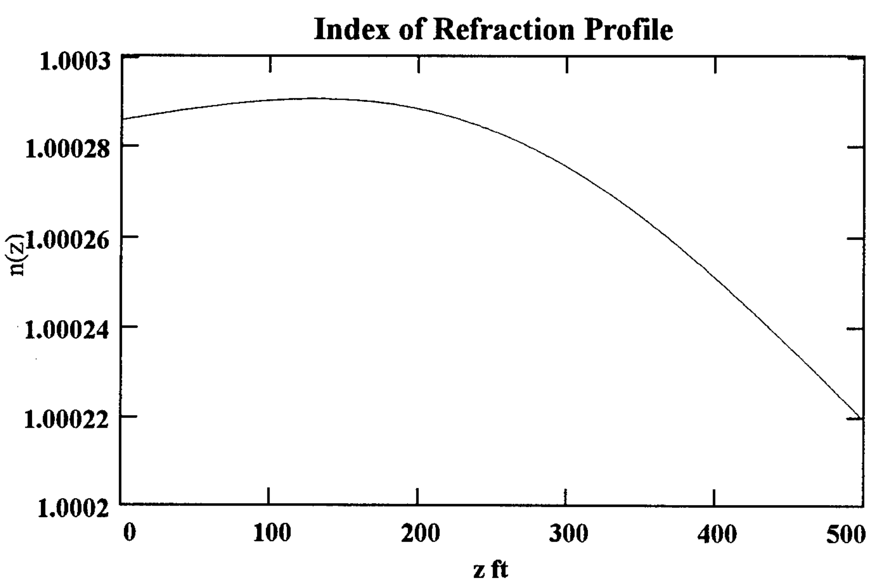
$$j := 0, 1 .. 45 \quad v_j := 10 \cdot j \quad w_j := M(v_j) \qquad u := 70 \quad x_e := \mathrm{root}(n(u) - n(0), u)$$

Write modified index profile to a file. $\mathbf{WRITEPRN(nprof0)} := \mathrm{augment}(v, w)$

The first and second derivatives n'(z), n"(z) are then given by:

$$n'(z) := 10^{-6} \cdot \left(\kappa - \frac{\Gamma + t'(z)}{t(z)} \cdot N(z)\right) \qquad \Gamma_3 := 3 \cdot \Gamma$$

$$n''(z) := 10^{-6} \cdot \left(\frac{2 \cdot t'(z)^2 + \Gamma_3 \cdot t'(z) - t(z) \cdot t''(z)}{t(z)^2}\right) \cdot N(z)$$



Raycalc3.mcd

Estimate the height where the index is a maximum and then calculate it. $u := 200$

$h_m := \mathrm{root}(n'(u), u)$ $h_m = 140.33556$ Avoid calculating a ray with a maximum at this height.

Choose the viewing eye height: $z_0 := 80$

$u := 400$ Ray maximum should be higher than $\mathrm{root}\left(n(u) - n\left(z_0\right), u\right) = 177.39753$

Calculate the height at which the index has the same value as at the ground. Actually choose .3 feet above the ground for this purpose to avoid complications in calculating the ray path.

$u := 500$ $h_e := \mathrm{root}(n(u) - n(.3), u) h_e = 229.76117$

Choose the height of the ray maximum: $z_m := 300$ $n_m := n\left(z_m\right)$ $n_2 := n_m{}^2$

$u := 1$ $\xi_0 := \mathrm{if}\left(z_m > h_e, 0, \mathrm{root}\left(n(u) - n_m, u\right)\right)$ $\xi_0 = 0$

Define some auxilliary functions: $\Phi(z) := \sqrt{\left|n(z)^2 - n_2\right|}$

Calculate the ray arclength from the viewing eye to the ray maximum.

$TOL := .1$ $z_{mn} := z_m - 1$

$$x_1 := n_m \cdot \int_{z_0}^{z_{mn}} \frac{1}{\Phi(\zeta)}\, d\zeta \quad x_2 := -n_m \cdot \frac{\Phi\left(z_{mn}\right)}{n\left(z_{mn}\right) \cdot n'\left(z_{mn}\right)} + n_m \cdot \int_{z_{mn}}^{z_m} \frac{n''(\zeta)}{\left(n(\zeta) \cdot n'(\zeta)\right)^2} \cdot \Phi(\zeta$$

$x_0 := x_1 + x_2$ $x_0 = 5.75122 \cdot 10^4$ $\xi_1 := \xi_0 + 1$

$$x_{00} := 2 \cdot x_0 - n_m \cdot \int_{z_0}^{\xi_1} \frac{1}{\Phi(\zeta)}\, d\zeta + n_m \cdot \frac{\Phi\left(\xi_1\right)}{n\left(\xi_1\right) \cdot n'\left(\xi_1\right)} - n_m \cdot \int_{\xi_1}^{\xi_0} \frac{n''(\zeta)}{\left(n(\zeta) \cdot n'(\zeta)\right)^2} \cdot \Phi(\zeta)\, d\zeta$$

$x_{00} = 2.15026 \cdot 10^5$ $\frac{x_0}{6080} = 9.45924$ $\frac{x_{00}}{6080} = 35.36618$

Define the second function used in the ray differential equations.

$h(x,z) := \mathrm{if}\left(x < x_0, \Phi(z), -\Phi(z)\right)$ $g(x,z) := \mathrm{if}\left(x < x_{00}, h(x,z), \Phi(z)\right)$

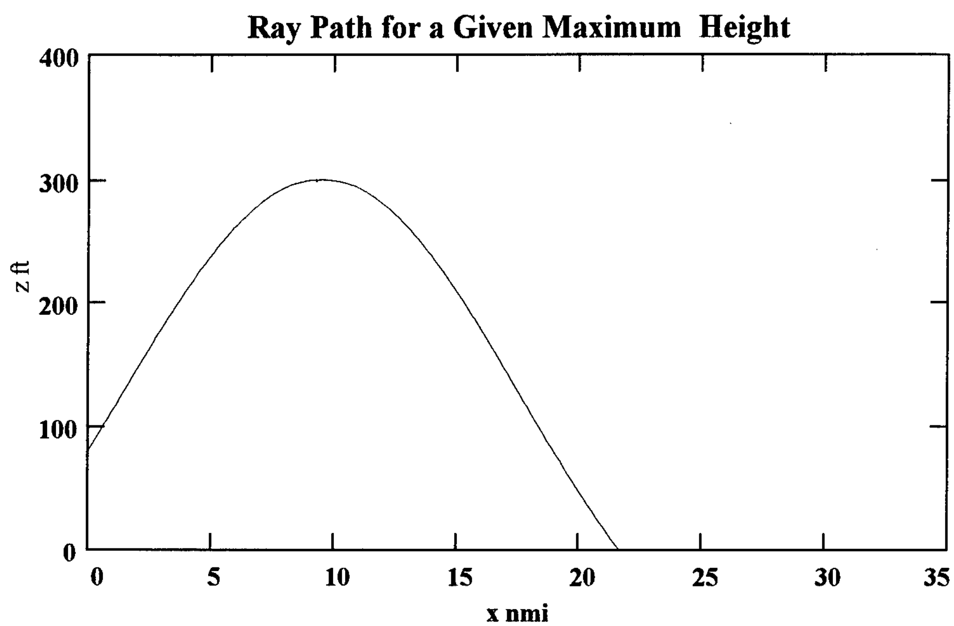
$f_2(x,z) := \mathrm{if}(z < 0, 0, g(x,z))$

Differential equations: $y_0 := z_0$ $D(x,y) := f_2\left(x, y_0\right)$

Solution of differential equations: $Z := \mathrm{rkfixed}(y, 0, 200000, 200, D)$

s = arclength, x = horizontal distance, z = vertical distance

$k := 0, 1 .. 200$ $\quad x_k := \dfrac{Z_{k,0}}{6080}$ $\quad z_k := \mathrm{if}\left(Z_{k,1} < 0, 0, Z_{k,1}\right)$



$Y := \mathrm{augment}(x, z)$ $\quad \mathrm{WRITEPRN}(ray8) := Y$ $\quad Y_{0,1} = 80$

To continue with another ray change the ray filename, go to page 4 and define a new ray height.

To plot all rays together first read in data from all ray files.

$U := \mathrm{READPRN}(ray1)$ $\quad V := \mathrm{READPRN}(ray2)$ $\quad W := \mathrm{READPRN}(ray3)$

$Y := \mathrm{READPRN}(ray4)$ $\quad Z := \mathrm{READPRN}(ray5)$ $\quad P := \mathrm{READPRN}(ray6)$

$Q := \mathrm{READPRN}(ray7)$ $\quad R := \mathrm{READPRN}(ray8)$ $\quad S := \mathrm{READPRN}(ray9)$

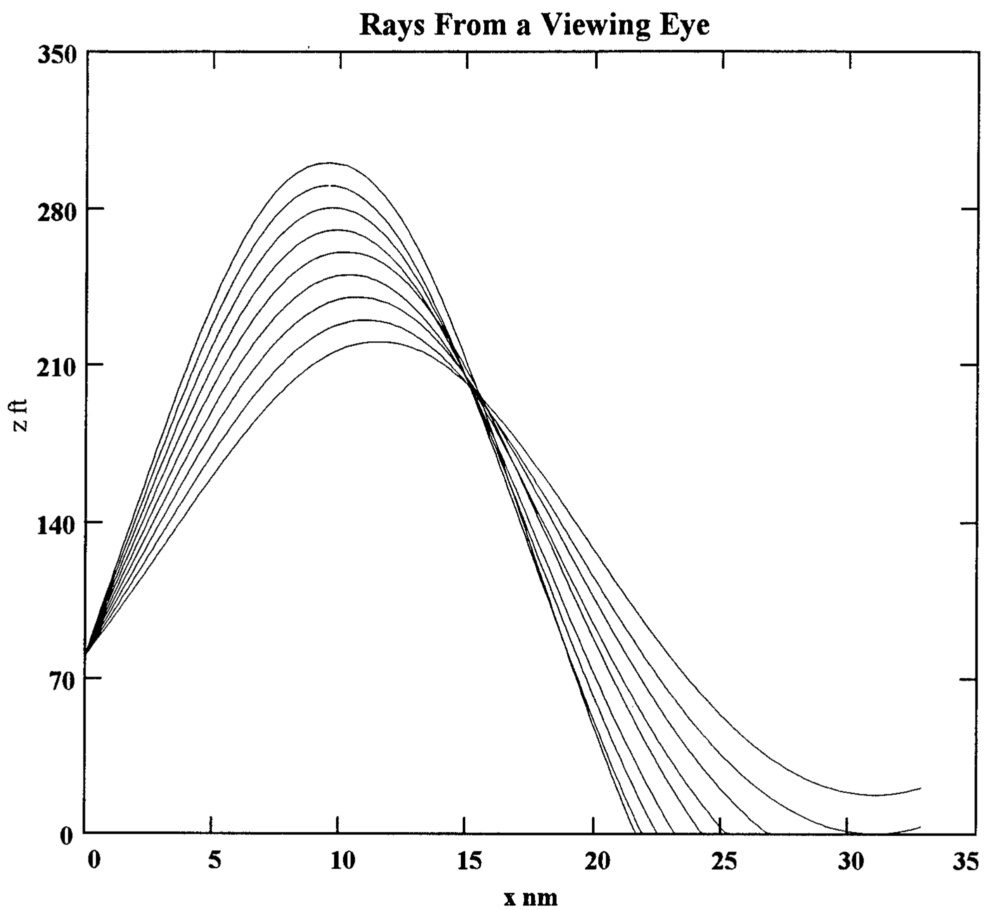
Assign a variable to the data for each ray. $\quad v := 200$ $\quad Y_{0,1} = 80$

$i := 0, 1 .. \nu$

$u_i := U_{i,1}$ $\quad v_i := V_{i,1}$ $\quad w_i := W_{i,1}$ $\quad y_i := Y_{i,1}$ $\quad z_i := Z_{i,1}$

$p_i := P_{i,1}$ $\quad q_i := Q_{i,1}$ $\quad r_i := R_{i,1}$ $\quad s_i := S_{i,1}$ $\quad x_i := S_{i,0}$

$y_0 = 80$



RAYCALC4.MCD

CAUTION! Click on Automatic in the Math menu to toggle to the manual mode before proceeding.

This program is specifically for rays in an inferior mirage, i.e., each with a minimum and a change in the sign of its curvature. The program generates a temperature profile and the corresponding index of refreaction profile. The function chosen for this purpose is a cubic polynomial that has only one real root and depends on three parameters and a scale factor sf. The program also traces a ray, given its minimum height, but it assumes a ray with at most one minimum and one maximum.

$\mathrm{TOL} := 10^{-8}$

$\mathrm{sf} := 100 \quad \mathrm{b} := 0 \quad \mathrm{c} := 2 \quad \mathrm{d} := 275 \quad \mathrm{a} := \mathrm{b}^2 + \mathrm{c}^2 \quad \mathrm{b2} := \frac{2 \cdot \mathrm{b}}{\mathrm{sf}^2} \quad \mathrm{sf3} := \mathrm{sf}^3 \quad \mathrm{a2} := -\frac{\mathrm{a}}{\mathrm{sf}}$

$\mathrm{t}(\mathrm{x}) := \frac{\mathrm{x}^3}{\mathrm{sf3}} - \mathrm{b2} \cdot \mathrm{x}^2 + \mathrm{a2} \cdot \mathrm{x} + \mathrm{d} \qquad \mathrm{f} := \frac{3}{\mathrm{sf3}} \qquad \mathrm{g} := 2 \cdot \mathrm{b2} \qquad \mathrm{h} := 2 \cdot \mathrm{f}$

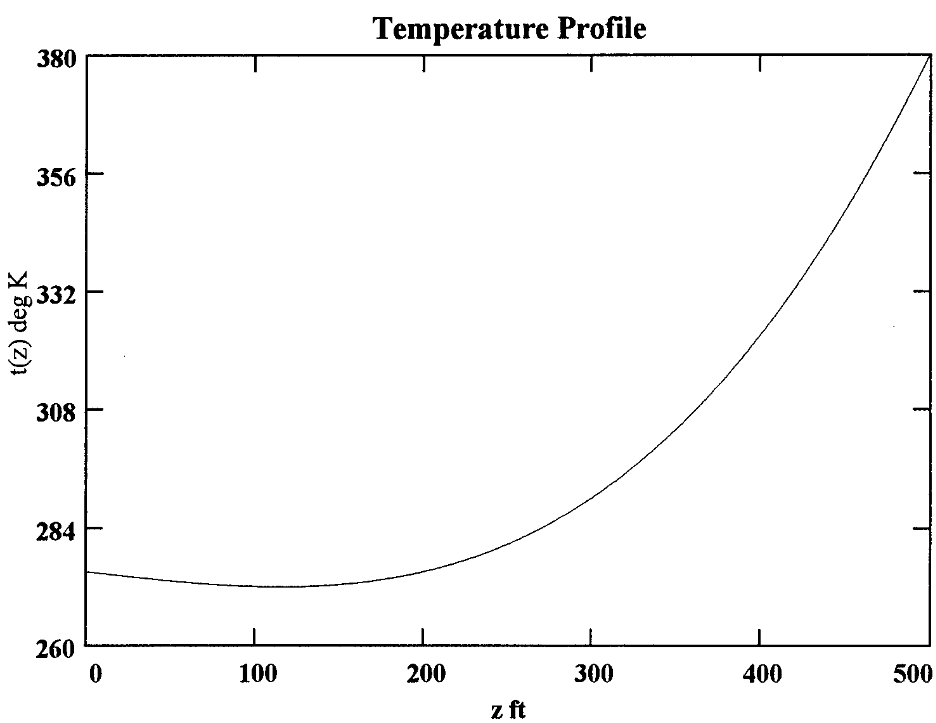
Check the lowest temperature, which is intitially in deg K., in deg C and deg F:

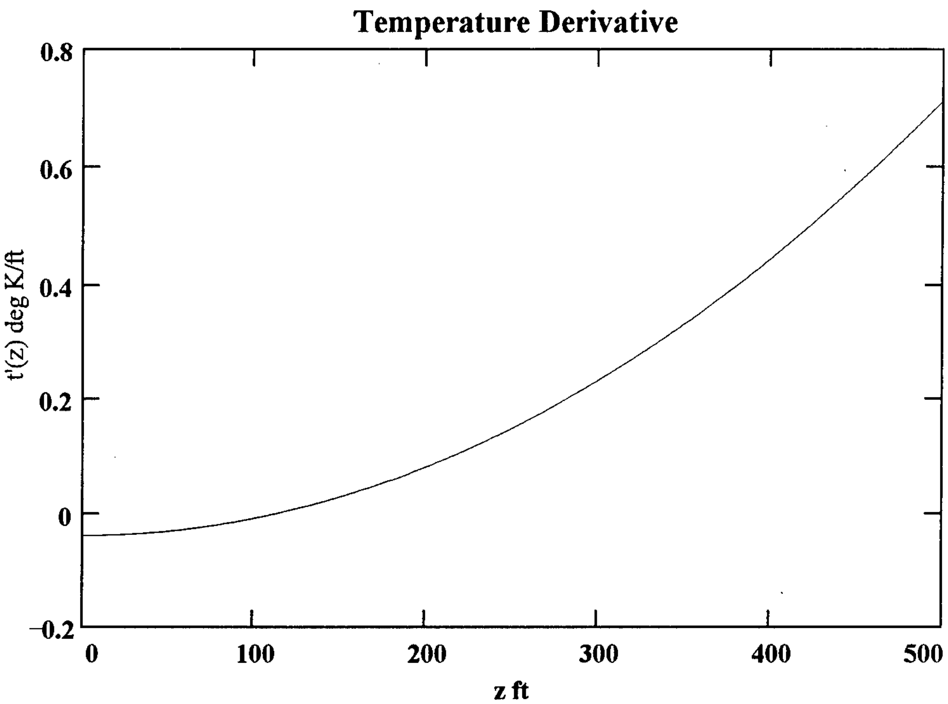
$\mathrm{C} := \mathrm{t}(0) - 273 \quad \mathrm{C} = 2 \qquad 9 \cdot \frac{\mathrm{C}}{5} + 32.2 = 35.8$

Get first and second derivatives t'(x) and t''(x) of the temperature profile.

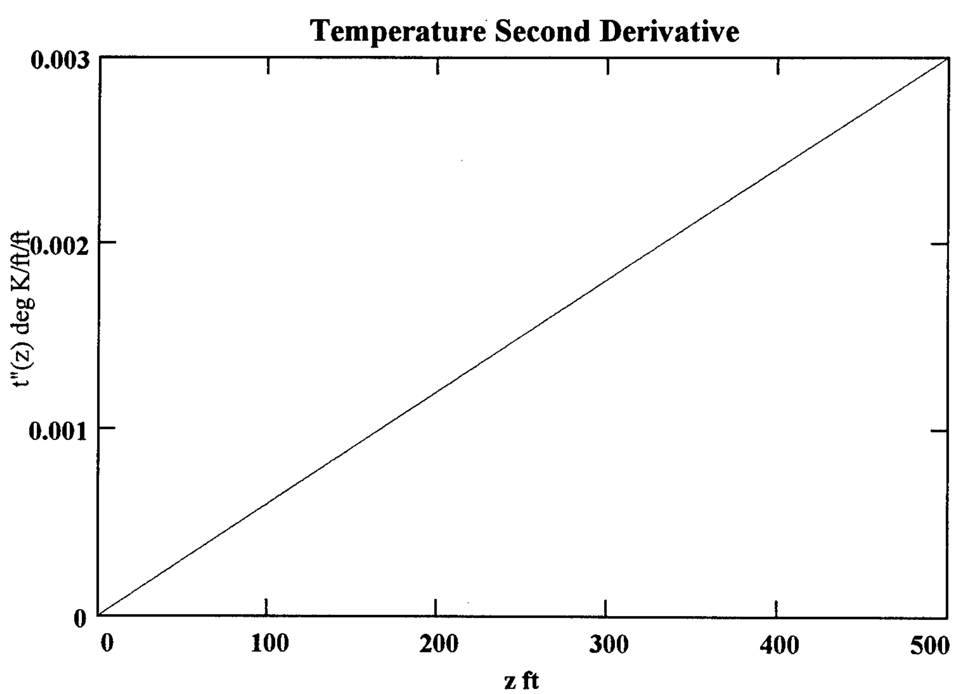
$\mathrm{t'}(\mathrm{x}) := \mathrm{f} \cdot \mathrm{x}^2 - \mathrm{g} \cdot \mathrm{x} + \mathrm{a2} \qquad \mathrm{t''}(\mathrm{x}) := \mathrm{h} \cdot \mathrm{x} - \mathrm{g}$

$\mathrm{z} := 0, 1 .. 500$





$x := 30$ $t_0 := \mathrm{root}(t'(x), x)$ $t_0 = 115.4700571$



Get the index of refraction profile. First define some universal constants: $N_0 := 77.5256$,

$\Gamma := .03417$, the standard atmospheric pressure at sea level $P_0 := 1013.25$, which, along

with the temperature profile t(z), determine the atomospheric pressure p(z) and the modulus of refraction N(z) as a function of the height z. The index of refraction n(z) is defined in terms of the modified modulus M(z). The parameter κ is .048 if the unit of height is feet or .157 if it is meters.

$$p(z) := P_0 \cdot e^{\left[-\Gamma \cdot \int_0^z \frac{1}{t(\zeta)} d\zeta\right]} \qquad N(z) := N_0 \cdot \frac{p(z)}{t(z)}$$

$\kappa := .048$ $\quad M(z) := N(z) + \kappa \cdot z \quad n(z) := 1 + 10^{-6} \cdot M(z)$

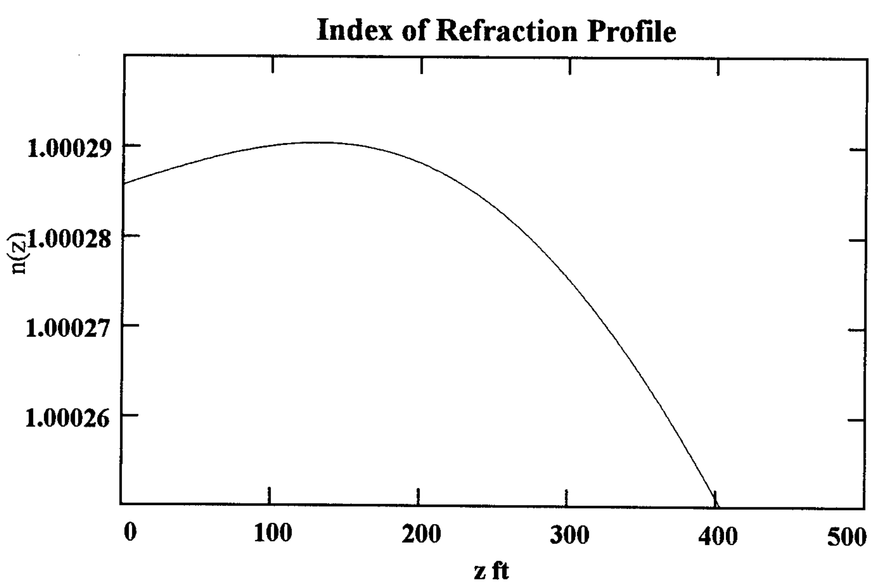
$j := 0, 1 .. 45 \quad v_j := 10 \cdot j \quad w_j := M(v_j)$

Write modified index profile to a file. $\quad \mathrm{WRITEPRN}(nprof0) := \mathrm{augment}(v, w)$

The first and second derivatives n'(z), n"(z) are then given by:

$$n'(z) := 10^{-6} \cdot \left(\kappa - \frac{\Gamma + t'(z)}{t(z)} \cdot N(z)\right) \qquad \Gamma_3 := 3 \cdot \Gamma$$

$$n''(z) := 10^{-6} \cdot \left(\frac{2 \cdot t'(z)^2 + \Gamma_3 \cdot t'(z) - t(z) \cdot t''(z)}{t(z)^2}\right) \cdot N(z)$$



$i := 0, 1 .. 50 \quad x_i := 10 \cdot i \quad t_i := t(x_i) \quad \mathrm{WRITEPRN}(temprof) := \mathrm{augment}(x, t)$

$v_i := n(x_i) \quad \mathrm{WRITEPRN}(nprof) := \mathrm{augment}(x, v)$

Estimate the height where the index is a maximum and then calculate it.

$u := 200 \quad h_m := \mathrm{root}(n'(u), u) h_m = 140.3355589$

Viewing eye height: $z_0 := 200 \quad u := 0 \quad \mathrm{root}(n(u) - n(z_0), u) = 49.2077951$

Choose a minimum height for a ray: $z_m := 25 \quad n_m := n(z_m) \quad n_2 := n_m^{\,2}$

Calculate the height where the index is the same.

$u := 500 \quad \xi_0 := \mathrm{root}(n(u) - n_m, u)$

$\xi_0 = 215.6876967 \quad \xi_1 := \xi_0 - 1$

Define some auxilliary functions: $\Phi(z) := \sqrt{\left| n(z)^2 - n_2 \right|}$

$TOL := .1$

Calculate the ray arclength from the viewing eye to the ray minimum.

$z_{mn} := z_m + 1$

$$x_1 := -n_m \cdot \int_{z_0}^{z_{mn}} \frac{1}{\Phi(\zeta)} \, d\zeta \quad x_2 := \frac{n_m \cdot \Phi(z_{mn})}{n(z_{mn}) \cdot n'(z_{mn})} - n_m \cdot \int_{z_{mn}}^{z_m} \frac{n''(\zeta)}{(n(\zeta) \cdot n'(\zeta))^2} \cdot \Phi(\zeta) \, d\zeta$$

$x_0 := x_1 + x_2 \quad x_0 = 9.8901947 \cdot 10^4$

$$x_{00} := 2 \cdot x_0 + n_m \cdot \int_{z_0}^{\xi_1} \frac{1}{\Phi(\zeta)} \, d\zeta - \frac{n_m \cdot \Phi(\xi_1)}{n(\xi_1) \cdot n'(\xi_1)} + n_m \cdot \int_{\xi_1}^{\xi_0} \frac{n''(\zeta)}{(n(\zeta) \cdot n'(\zeta))^2} \cdot \Phi(\zeta) \, d\zeta$$

$x_{00} = 2.1728195 \cdot 10^5$

$x_{000} := 2 \cdot x_{00} - x_0 \quad x_{000} = 3.3566196 \cdot 10^5$

Define the second function used in the ray differential equations.

$$h(x,z) := \mathrm{if}\left(x<x_0, -\Phi(z), \Phi(z)\right) \qquad g_1(x,z) := \mathrm{if}\left(x>x_{00}, -\Phi(z), h(x,z)\right)$$

$$f_1(x,z) := \mathrm{if}\left(x>x_{000}, \Phi(z), g_1(x,z)\right)$$

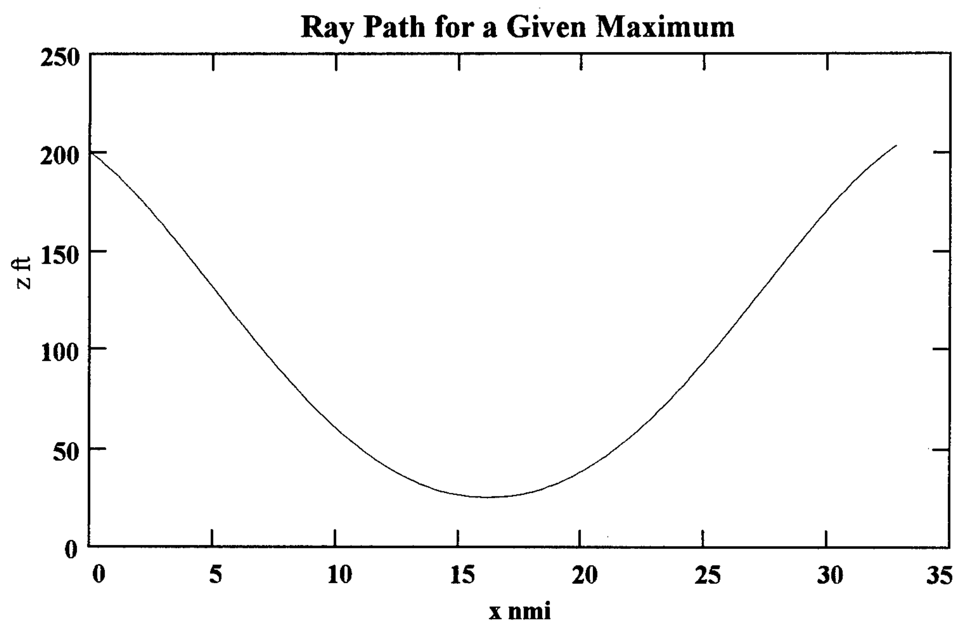
$$f(x,z) := \mathrm{if}\left(z<0, 0, f_1(x,z)\right)$$

Differential equations: $y_0 := z_0$ $\qquad D(x,y) := f\left(x, y_0\right)$

Solution of differential equations: $Z := \mathrm{rkfixed}(y, 0, 200000, 200, D)$

s = arclength, x = horizontal distance, z = vertical distance

$k := 0, 1 .. 200 \qquad x_k := \dfrac{Z_{k,0}}{6080} \qquad z_k := \mathrm{if}\left(Z_{k,1}<0, 0, Z_{k,1}\right)$



$Y := \mathrm{augment}(x, z) \qquad \mathrm{WRITEPRN}(ray9) := Y$

To continue with another ray change the ray filename, go to page 4, and define a new ray height.

To plot all rays together first read in data from all ray files.

$\mathbf{U} := \mathbf{READPRN(ray1)}$ $\mathbf{V} := \mathbf{READPRN(ray2)}$ $\mathbf{W} := \mathbf{READPRN(ray3)}$

$\mathbf{Y} := \mathbf{READPRN(ray4)}$ $\mathbf{Z} := \mathbf{READPRN(ray5)}$ $\mathbf{P} := \mathbf{READPRN(ray6)}$

$\mathbf{Q} := \mathbf{READPRN(ray7)}$ $\mathbf{R} := \mathbf{READPRN(ray8)}$ $\mathbf{S} := \mathbf{READPRN(ray9)}$

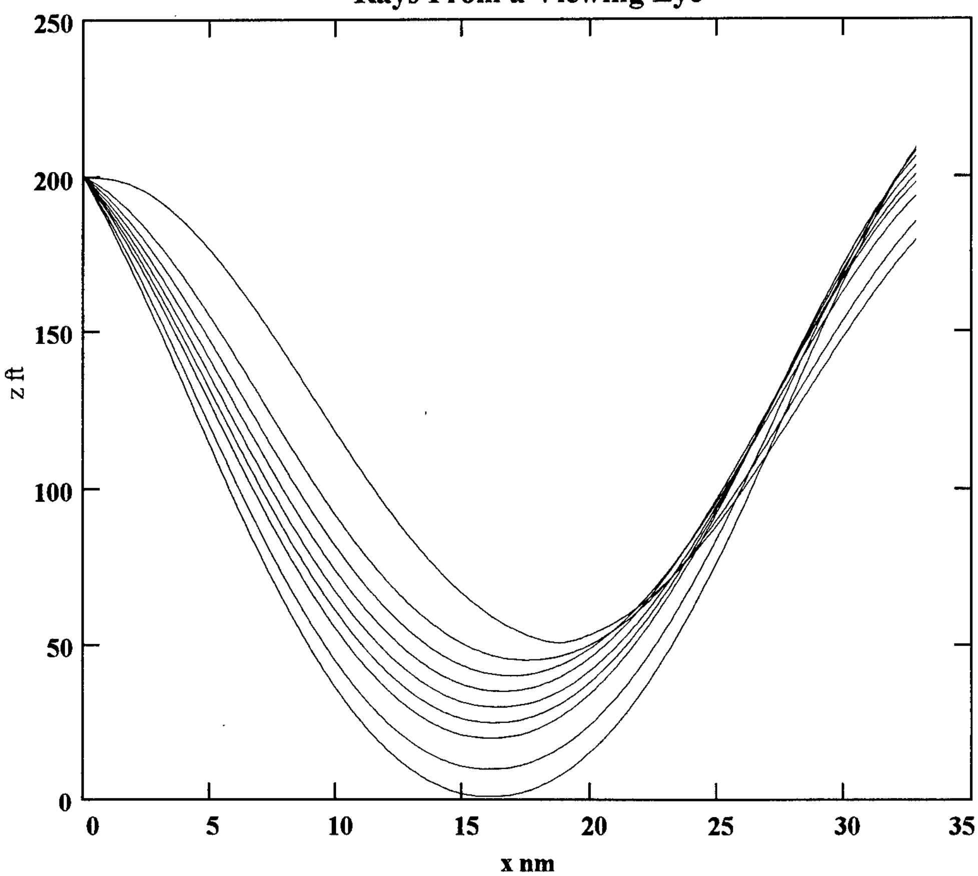
Assign a variable to the data for each ray. $\nu := 200$

$i := 0, 1 .. \nu$

$u_i := U_{i,1}$ $v_i := V_{i,1}$ $w_i := W_{i,1}$ $y_i := Y_{i,1}$ $z_i := Z_{i,1}$

$p_i := P_{i,1}$ $q_i := Q_{i,1}$ $r_i := R_{i,1}$ $s_i := S_{i,1}$ $x_i := S_{i,0}$

Rays From a Viewing Eye



REPORT DOCUMENTATION PAGE

Form Approved OMB No. 0704-0188

1. AGENCY USE ONLY *(Leave blank)*	2. REPORT DATE	3. REPORT TYPE AND DATES COVERED
	May 1995	Final—November 1994–May 1995

4. TITLE AND SUBTITLE
Atmospheric Ray Tracing for Predicting Mirages

5. FUNDING NUMBERS
DASW01 94 C 0054
ARPA Assignment A-180

6. AUTHOR(S)
Irvin W. Kay

7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)
Institute for Defense Analyses
1801 N. Beauregard St.
Alexandria, VA 22311-1772

8. PERFORMING ORGANIZATION REPORT NUMBER
IDA Document D-1692

9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES)
Advanced Research Projects Agency
3701 N. Fairfax Drive
Arlington, VA 22203-1714

10. SPONSORING/MONITORING AGENCY REPORT NUMBER

11. SUPPLEMENTARY NOTES

12a. DISTRIBUTION/AVAILABILITY STATEMENT
Approved for Public Release, Distribution Unlimited.

12b. DISTRIBUTION CODE

13. ABSTRACT *(Maximum 180 words)*

At infrared-optical frequencies the Edlen formula determines the index of refraction profile in a continuous, spherically symmetric atmosphere in terms of the pressure and temperature. Since the pressure dependence is known in terms of the temperature when equilibrium exists, a given temperature profile determines that of the index of refraction. The classical equation for rays propagating in such a medium then provides various geometrical properties of a ray, given its nearest minimum or maximum height relative to a specified viewing eye through which it passes. For altitudes above the surface of the Earth that are small compared to its radius the propagation geometry can be approximated in terms of a plane stratified medium above a flat ground surface. However, for distances large enough to include a horizon ray, a single corresponding approximate index of refraction profile is only valid for predicting the behavior of nearby rays. The slope of the index of refraction profile as well as that of the temperature determine the sign of the curvature of any ray at any height. Using the relationship between geometrical properties of the index of refraction profile it is possible to draw qualitative conclusions about the nature of possible mirages.

14. SUBJECT TERMS
mirage, ray propagation, refraction profile, Edlen formula

15. NUMBER OF PAGES
40

16. PRICE CODE

17. SECURITY CLASSIFICATION OF REPORT	18. SECURITY CLASSIFICATION OF THIS PAGE	19. SECURITY CLASSIFICATION OF ABSTRACT	20. LIMITATION OF ABSTRACT
UNCLASSIFIED	UNCLASSIFIED	UNCLASSIFIED	SAR